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A NOTE ON PLANE KINEMATICS.

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INTRODUCTION.

It appears from the nature of the subject that the use of vector methods must be of advantage in studying the motion of a rigid body. A. FÖPPL'S *Technische Mechanik,* F. CASTELLANO'S *Meccanica razionale* (1894), and especially R. MARCOLONGO'S *Meccanica razionale* (Milano, Hoepli, 1905, 2 vols., German edition by TIMERDING, Teubner, 1911–12) bear evidence to this effect.

The present paper is confined to *plane* motion where the number of new symbols is very small. The notation used is that of C. BURALI-FORTI et R. MARCOLONGO, *Elêments de calcul vectoriel,* Paris, Hermann, 1910.

The vector from the point O to the point P is denoted by $P - O$, or by a single bold-faced letter such as $\boldsymbol{p}$. The time-derivative of a moving point P is denoted by a dot; thus, if P, P' are the positions of the point at the times t, $t + \Delta t$, we have

$$\dot{P} = \lim_{\Delta t = 0} \frac{P' - P}{\Delta t}$$

for the *velocity* of P. The *acceleration* of P is the second time-derivative $\ddot{P}$. The time-derivative of the vector $\boldsymbol{p} = P - O$ is $\dot{\boldsymbol{p}} = \dot{P} - \dot{O}$; this is the velocity of P *relative* to O.

The absolute value, or length, of a vector $\boldsymbol{p} = P - O$ is indicated by mod $\boldsymbol{p}$, or mod $(P - O)$. The scalar product of two vectors $\boldsymbol{a}$, $\boldsymbol{b}$, i. e., the product of their lengths into the cosine of the angle of their directions, is written $\boldsymbol{a} \times \boldsymbol{b}$; for $\boldsymbol{a} \times \boldsymbol{a}$ we write $\boldsymbol{a}^2$. The vector product $\boldsymbol{a} \wedge \boldsymbol{b}$ (read $\boldsymbol{a}$ vec $\boldsymbol{b}$), which is the vector at right angles to both $\boldsymbol{a}$ and $\boldsymbol{b}$, representing by its length the area of the parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$, is not required in the plane.

Rotation through a right angle in the positive (say, counter-clockwise) sense is indicated by i; thus $i\boldsymbol{a}$, $i\dot{P}$ are the vectors obtained by turning $\boldsymbol{a}$, $\dot{P}$ through the angle $+\frac{1}{2}\pi$. Finally, the vector obtained by turning the vector $\boldsymbol{a}$ through the angle φ is written $e^{\phi i}\boldsymbol{a}$.

These are all the special notations required in what follows.

Most of the results obtained are of course well known; but it is believed that the constructions of Arts. 6, 7, 11–16 are new.

I. VELOCITIES.

1. If O, P are any two different points of the moving figure and we put $P - O = \boldsymbol{p}$, the condition of rigidity $\boldsymbol{p}^2 =$ const. gives upon differentiation with respect to the time:

$$\dot{\boldsymbol{p}} \times \boldsymbol{p} = 0, \quad \text{or} \quad (\dot{P} - \dot{O}) \times (P - O) = 0;$$

i. e., *the relative velocity* $\dot{\boldsymbol{p}} = \dot{P} - \dot{O}$ *of any point with respect to any other point of the figure is perpendicular to the line joining the points* (Fig. 1).

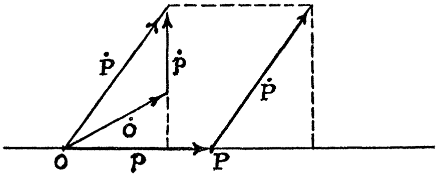

FIG. 1.

In other words, the projections of the velocities $\dot{O}$, $\dot{P}$ of any two points O, P on OP are equal in magnitude and sense.

2. The relation of the last article can be written in the form

$$\dot{\boldsymbol{p}} = wi\boldsymbol{p}, \quad \text{or} \quad \dot{P} - \dot{O} = wi(P - O), \tag{1}$$

where w is a (positive or negative) scalar.

This scalar is independent of the point P. For if Q be any other point of the figure and we put $Q - O = \boldsymbol{q}$ we have the additional conditions of rigidity $\boldsymbol{q}^2 =$ const., $\boldsymbol{p} \times \boldsymbol{q} =$ const., which give upon differentiation:

$$\dot{\boldsymbol{q}} \times \boldsymbol{q} = 0, \qquad \dot{\boldsymbol{p}} \times \boldsymbol{q} + \boldsymbol{p} \times \dot{\boldsymbol{q}} = 0.$$

Writing the former equation in the form $\dot{\boldsymbol{q}} = w_1 i\boldsymbol{q}$ and substituting in the latter equation for $\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{q}}$ their values we obtain (since $\boldsymbol{a} \times i\boldsymbol{b} = -i\boldsymbol{a} \times \boldsymbol{b}$):

$$(w - w_1)i\boldsymbol{p} \times \boldsymbol{q} = 0.$$

This gives $w_1 = w$ unless $i\boldsymbol{p}$ is perpendicular to $\boldsymbol{q}$, i. e., unless O, P, Q are collinear. But in this case we have $\boldsymbol{q} = m\boldsymbol{p}$ where m is a constant scalar; hence $\dot{\boldsymbol{q}} = m\dot{\boldsymbol{p}}$, which gives again $w_1 = w$.

3. If $w = 0$ it appears from (1) that the velocities of any two points, and hence the velocities of all points, of the figure are equal. This case presents no further interest.

If $w \neq 0$ equation (1) shows that the velocities of all points P relative to any one point O are at right angles to OP and proportional to the distance OP. For this reason the scalar w is called the ANGULAR VELOCITY of the plane motion.

4. If $w \neq 0$ we can find a point of the figure whose velocity is instantaneously zero. For if this point be called C we have from (1) the condition $0-\dot{O}=wi(C-O)$; dividing by w and multiplying by i we find:

$$C = O + \frac{1}{w} i\dot{O}. \tag{2}$$

There exists only one such point. For, dividing (1) by w and multiplying it by i we have

$$\frac{1}{w} i\dot{P} - \frac{1}{w} i\dot{O} = -P + O,$$

whence

$$P + \frac{1}{w} i\dot{P} = O + \frac{1}{w} i\dot{O} = C,$$

by (2); i. e., the same point C is reached from whichever point P of the figure we may start.

This point C of zero velocity is called the INSTANTANEOUS CENTER of the plane motion.

5. If the velocity $\dot{O}$ of any point O and the angular velocity w are given, the instantaneous center C is found, according to (2), by turning $\dot{O}$ about O through $\pi/2$ and multiplying it by $1/w$.

It follows that C lies on the normal of the path of every point of the figure and hence can also be found as the intersection of the normals to the paths of any two points whose velocities are not parallel (Fig. 2).

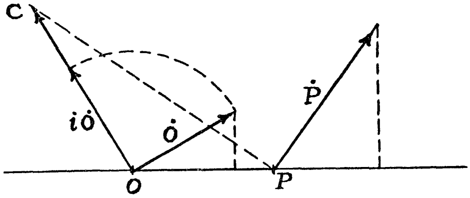

FIG. 2.

If in (1) we take for the arbitrary point O the instantaneous center C whose velocity is zero we find:

$$\dot{P} = wi(P - C); \tag{3}$$

i. e., the velocity of every point P of the figure is perpendicular to its radius vector drawn from C and proportional to its distance from C; in other words, *if not a translation* (all velocities equal, $w = 0$), *the instantaneous motion of a plane rigid figure in its plane is a rotation about the instantaneous center, with angular velocity w.*

6. If the velocities $\dot{O}, \dot{P}$ of any two points O, P of the figure are given (subject to the condition of Art. 1), the velocity $\dot{Q}$ of any other point Q of the figure can be found without constructing the instantaneous center. For, the fundamental relation (1) shows that if the vectors $\dot{O}, \dot{P}$ be drawn from the same origin, say O (Fig. 3), the line-segment $O'P'$ joining their extremities is perpendicular to OP and w times OP. It follows that the extremities of the velocities of any number of points of the figure, if drawn from the same origin, will form a figure similar to that of the points (though turned through a right angle), the ratio of similarity being w.

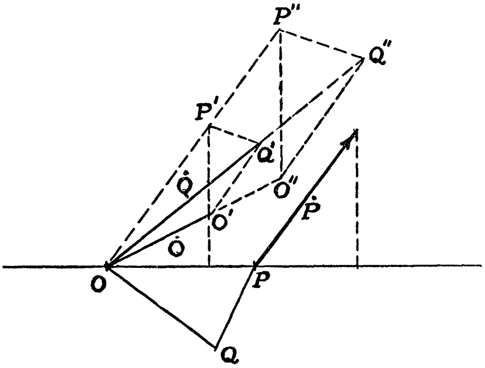

FIG. 3.

Hence, to find $\dot{Q}$ from $\dot{O}$ and $\dot{P}$ make (Fig. 3) $O' - O = \dot{O}$, $P' - O = \dot{P}$, and construct on $O'P'$ a triangle $O'P'Q'$ similar to OPQ, by drawing $O'Q'$ at right angles to OQ and $P'Q'$ at right angles to PQ; then $Q' - O = \dot{Q}$.

If we had used for the velocities a unit $= 1/w$ times the unit of length, making (Fig. 3) $O'' - O = (1/w)\dot{O}$, $P'' - O = (1/w)\dot{P}$, we should have $O''P'' = OP$ and triangle $O''P''Q''$ congruent to triangle OPQ; O is the center of similarity of the triangles $O'P'Q'$ and $O''P''Q''$.

7. Another method for deriving $\dot{Q}$ from $\dot{O}$ and $\dot{P}$ consists in using projections, according to the last statement of Art. 1. The projections of the required velocity $\dot{Q}$ on OQ and PQ must be equal, respectively, to the projections of $\dot{O}$ on OQ and

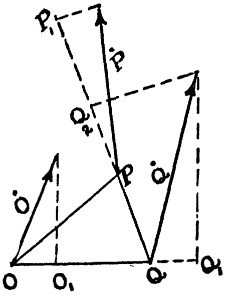

FIG. 4.

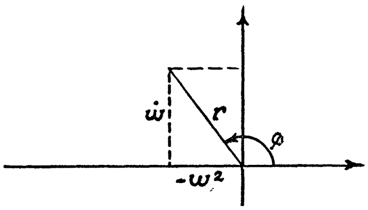

FIG. 5.

of P on PQ; and from these known projections Q can be found (Fig. 4): make $QQ_1 = OO_1$, $QQ_2 = PP_1$, and draw through Q_1 a perpendicular to OQ, through Q_2 a perpendicular to PQ.

II. Accelerations.

8. The fundamental relation (1) gives upon differentiation with respect to the time:

$$\ddot{\mathbf{p}} = wi\dot{\mathbf{p}} + \dot{w}i\mathbf{p} = -w^2\mathbf{p} + \dot{w}i\mathbf{p}, \tag{4}$$

i. e.,

$$\ddot{P} = \ddot{O} + (-w^2 + \dot{w}i)(P - O).$$

Let us write the complex number $-w^2 + \dot{w}i$ in the polar form (Fig. 5):

$$-w^2 + \dot{w}i = re^{\phi i},$$

so that

$$r = \sqrt{w^4 + \dot{w}^2}, \qquad \tan\varphi = \frac{\dot{w}}{-w^2}.$$

Taking r positive we have $\frac{1}{2}\pi \leqq \varphi \leqq \frac{3}{2}\pi$ and $\varphi \lesseqgtr \pi$ according as $\dot{w} \gtreqless 0$. The case of translation when both w and $\dot{w}$ are zero is here excluded.

We then have simply

$$\ddot{\mathbf{p}} = re^{\phi i}\mathbf{p}, \quad \text{or} \quad \ddot{P} = \ddot{O} + re^{\phi i}(P - O); \tag{4'}$$

i. e., *the acceleration* $\ddot{P}$ *of any point* P *is found from the acceleration* $\ddot{O}$ *of any other point* O *by adding to* $\ddot{O}$ *the vector obtained by turning* $P - O$ *through the angle* φ *and multiplying it by* r, φ and r being, at a given instant, the same for all points P of the figure.

9. As $r \neq 0$ (the case of translation being excluded) we can always find one and only one point, say A, whose acceleration is zero. For, the condition $0 = \ddot{O} + re^{\phi i}(A - O)$ gives

$$A = O - \frac{1}{r}e^{-\phi i}\ddot{O}. \tag{5}$$

This point A of zero acceleration is called the CENTER OF ACCELERATION.

If in the general relation (4′) we select for O the center of acceleration we find

$$\ddot{P} = re^{\phi i}(P - A); \tag{6}$$

i. e., *the acceleration* $\ddot{P}$ *of any point* P *is obtained by turning the radius vector of* P, *drawn from* A, *through the angle* φ *and multiplying it by* r.

The acceleration $\ddot{P}$ has therefore two components: one, w^2 mod $(P - A)$, directed toward A, the other, $\dot{w}$ mod $(P - A)$, at right angles to AP.

All points of a circle about A have accelerations of equal magnitude and equally inclined to their radii vectores drawn from A; all points of a line through A have parallel accelerations, in magnitude proportional to the distance from A.

10. If, in particular, $\dot{w} = 0$ we have (Art. 8) $r = w^2$, $\varphi = \pi$, so that (5) and (6) reduce to

$$A = O + \frac{1}{w^2}\ddot{O}, \qquad \ddot{P} = -\,w^2(P - A).$$

Thus, *if the angular velocity* w *is constant (but* $\neq 0$*), the accelerations of all points are directed toward one and the same point, the center of acceleration, and are proportional to the distance from this point.*

If, on the other hand, $w = 0$ while $\dot{w} \neq 0$, we have if $\dot{w}$ is positive: $r = \dot{w}$, $\varphi = \frac{1}{2}\pi$, and if $\dot{w}$ is negative: $r = -\,\dot{w}$, $\varphi = \frac{3}{2}\pi$, so that (5) and (6) reduce to

$$A = O + \frac{1}{\dot{w}}i\ddot{O}, \qquad \ddot{P} = \dot{w}i(P - A).$$

Thus, *if the angular velocity* w *is zero (while* $\dot{w} \neq 0$*), the accelerations of all points are at right angles to their radii vectores drawn from the center of acceleration and proportional to the distance from it.*

11. Comparing the equations (1) and (4′) it appears that the field of acceleration vectors, like that of velocity vectors, is completely determined by any two vectors. But in the case of the velocities the vectors $\dot{O}$, $\dot{P}$ cannot be selected arbitrarily since (Art. 1) $\dot{\mathbf{p}} \times \mathbf{p} = 0$, i. e., $\dot{O}$, $\dot{P}$ must have equal projections on OP. In the case of the accelerations the vectors $\ddot{O}$, $\ddot{P}$ can be taken arbitrarily.

Multiplying (4) by $\times \mathbf{p}$ we find

$$\ddot{\mathbf{p}} \times \mathbf{p} = -\,w^2\mathbf{p}^2; \tag{7}$$

i. e., *the projections of the accelerations* $\ddot{O}$, $\ddot{P}$ *of any two points* O, P *of the figure on the line* OP *differ by a quantity (*w^2 *mod* $\mathbf{p}$*) proportional to the distance* OP *of the points.*

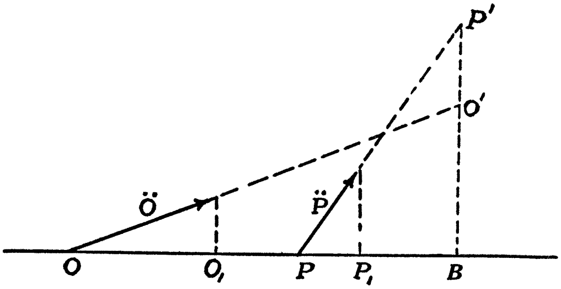

FIG. 6.

To exhibit this relation graphically let us take the unit of acceleration as $1/w^2$ times the unit of length; the difference of the projections of the acceleration vectors, drawn on this scale, will then be equal to the distance OP; i. e., if these vectors be drawn from O and P, the projections of their extremities on OP will coincide, say at B.

Thus, if OO_1, PP_1 (Fig. 6) are the projections on OP of the actual accelerations (drawn on the same scale as the distances), and if $O' - O = (1/w^2)\ddot{O}$, $P' - P = (1/w^2)\ddot{P}$, the projections of O' and P' on OP coincide at B. As OB is divided by O_1 in the same ratio (viz., w^2) in which PB is divided by P_1, we have

$$\frac{OB}{OO_1} = \frac{OB - OP}{PP_1}, \qquad \text{whence} \qquad OB = \frac{OO_1}{OO_1 - PP_1} \cdot OP,$$

or putting $PP_1/OO_1 = m$:

$$OB = \frac{1}{1 - m} OP.$$

12. This leads to a simple *construction of the center of acceleration A.* Indeed, A is the intersection (different from B) of the circles described about OO' and PP' as diameters (Fig. 7).

For, any point Q on the circle about OO' has its acceleration at right angles to OQ since the projection of OO' on OQ is OQ; and similarly any point R of the circle about PP' has its acceleration at right angles to PR. Hence the acceleration of A, which would have to be perpendicular to both OA and PA, must be zero.

13. The acceleration of O, if represented by $O' - O$, has the rectangular components $A - O$ and $O' - A$; similarly the acceleration $P' - P$ of P has the rectangular components $A - P$ and $P' - A$. Now (Fig. 7) $\sphericalangle OO'A = OBA$, $\sphericalangle OBA = PBA = PP'A$; hence $\sphericalangle OO'A = PP'A$. The rightangled triangles OAO', PAP' are therefore similar, and hence

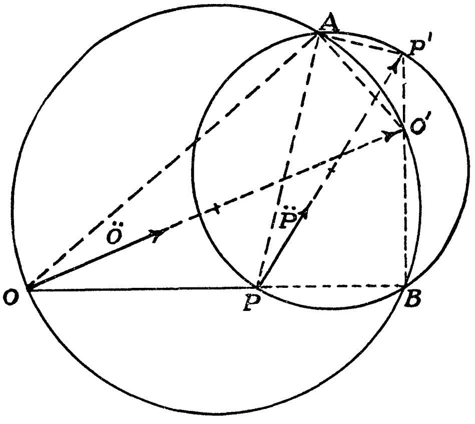

FIG. 7.

$$\frac{OA}{AO'} = \frac{PA}{AP'}.$$

This means that if the acceleration of any point be resolved along and at right angles to the radius vector drawn from the center of acceleration, these components have the same ratio for all points of the figure. This, of course, also follows from formula (6) since $\sphericalangle AOO'$ is the supplement of φ.

The above construction of the center of acceleration follows also from formula (5) which can be written:

$$A = O + \frac{1}{r}e^{(\pi-\phi)i}\ddot{O} = O + \frac{w^2}{r}e^{(\pi-\phi)i}\cdot\frac{1}{w^2}\ddot{O},$$

or since $w^2/r = -\cos\varphi$(Art. 8):

$$A = O - \cos\varphi\cdot e^{(\pi-\phi)i}\cdot\frac{1}{w^2}\ddot{O}.$$

14. It follows from the preceding articles that *if the accelerations $\ddot{O}$, $\ddot{P}$ of any two points O, P of the figure are given, the acceleration $\ddot{Q}$ of any other point Q can be found,* provided $w \neq 0$. For, by Art. 11 we can find the point B, and from this by Art. 12 the center of acceleration A; hence, constructing on AQ a triangle AQQ' similar to AOO' we find

$$\frac{1}{w^2}\ddot{Q} = Q' - Q, \quad \text{where} \quad w^2 = \frac{\operatorname{mod}\ddot{O}}{\operatorname{mod}(O' - O)}.$$

This construction cannot be used when $w = 0$. But in this case (Art. 10) A is found as the intersection of perpendiculars through O, P to $\ddot{O}$, $\ddot{P}$, respectively. If these perpendiculars happen to be parallel, A lies at infinity; if they coincide, A is found as the intersection of OP with the line joining the extremities of $\ddot{O}$ and $\ddot{P}$.

15. The acceleration $\ddot{Q}$ of any point Q of the figure can however be found from the accelerations $\ddot{O}$, $\ddot{P}$ of any two points O, P without first constructing the center of acceleration. The construction follows directly from the relations $\ddot{\mathbf{p}} = re^{\phi i}\mathbf{p}$, $\ddot{\mathbf{q}} = re^{\phi i}\mathbf{q}$, where $\mathbf{q} = Q - O$ (Fig. 8). The vectors $P - O = \mathbf{p}$ and $\ddot{P} - \ddot{O} = \ddot{\mathbf{p}}$,

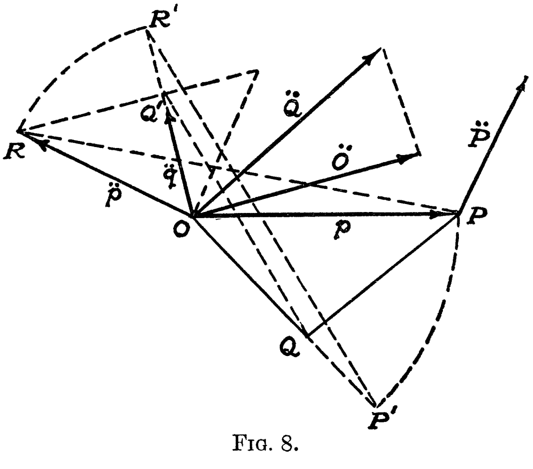

FIG. 8.

drawn from O, form a triangle OPR with the angle φ at O. Turn this triangle about O through an angle $= \measuredangle POQ$ so that it takes the position $OP'R'$; a parallel through Q to $P'R'$ then determines $\ddot{q} = Q' - O$. Adding this to $\ddot{O}$ we find $\ddot{Q}$.

16. Finally, by using projections somewhat as in Art. 7 for velocities, we can derive $\ddot{Q}$ from $\ddot{O}$ and $\ddot{P}$ as follows.

Let OO_1 and OP_1 be the projections on OP of $\ddot{O}$ and $\ddot{P}$, drawn from O (Fig. 9), and construct on P_1O_1 the triangle $P_1O_1Q_1$ similar to OPQ. Let the parallel to OP through Q_1 meet OQ at Q', PQ at Q''; and let OO' be the projection of $\ddot{O}$ on OQ, PP' the projection of $\ddot{P}$ on PQ.

Then $Q'O'$ is the projection of $\ddot{O}$ on QO, $Q''P'$ the projection of $\ddot{Q}$ on QP; transferring these to Q, $\ddot{Q}$ is found.

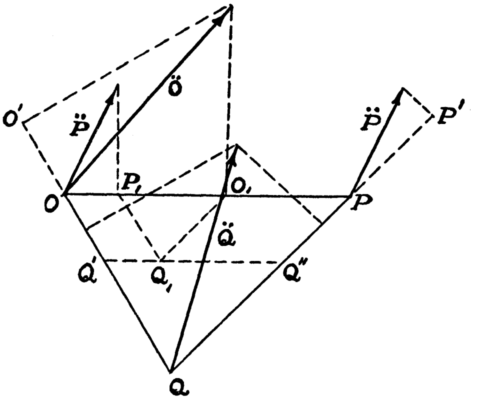

FIG. 9.

The constructions of Arts. 7, 11–13, 16 were first obtained from the equations in Cartesian coordinates, OP being taken as axis of x. Using the vector method it is naturally preferable to deal with the vectors themselves rather than with their projections.

NOTE ON CERTAIN ALGEBRAIC EQUATIONS.

By HERMON L. SLOBIN, University of Minnesota.

In elementary analysis we often meet problems like the following:

Show that the roots of the equation

I. $$x^3 + x^2 - 2x - 1 = 0 \quad \text{are} \quad 2\cos\frac{2}{7}\pi, \quad 2\cos\frac{4}{7}\pi, \quad 2\cos\frac{8}{7}\pi.$$

Show that the roots of the equation

II. $$x^3 - 5x^2 + 6x - 1 = 0 \quad \text{are} \quad 4\cos^2\frac{\pi}{7}, \quad 4\cos^2\frac{2\pi}{7}, \quad 4\cos^2\frac{4\pi}{7}.$$